ERRATUM: "AN INFINITE PRESENTATION FOR THE MAPPING CLASS GROUP OF A NON-ORIENTABLE SURFACE WITH BOUNDARY"

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The purpose of this note is to call attention to an error in the paper [1]. In addition, we notice the abelisnization of the mapping class group $\mathcal{M}(N_{g,n})$ for $g = 1, 2$.

Correction of relations. There is an error in part of the relations (D1a)–(D4g) in the presentation in [1, Proposition 3.2]. The mistake is due to a miscalculation of the exponent of d_k[1] at the end of the relations (we call the exponent the *index*). We list the revision of relations as follows:

- Relation (D1b) for $i = 1$:
  "$y(a_{i;k}a_i^{-1})y^{-1} = \dots\ r_{2;k}r_{1;k}d_{n-1}^{-2}$" $\to$ "$y(a_{i;k}a_i^{-1})y^{-1} = \dots\ r_{2;k}r_{1;k}$",
- Relation (D2a) for $m = i - 1$:
  "$a_m r_{i;k} a_m^{-1} = \dots\ r_{i;k}(a_{i-1;k}a_{i-1}^{-1})d_k$" $\to$ "$a_m r_{i;k} a_m^{-1} = \dots\ r_{i;k}(a_{i-1;k}a_{i-1}^{-1})$",
- Relation (D2b) for $i = 2$:
  "$y r_{i;k} y^{-1} = \dots\ r_{2;k}r_{1;k}d_{n-1}$" $\to$ "$y r_{i;k} y^{-1} = \dots\ r_{2;k}r_{1;k}$",
- Relation (D4e) for $m = 1$ and $l > j$:
  "$r_{m;l}(\bar{s}_{j;k}d_j^{-1})r_{m;l}^{-1} = \{\ \dots\ \}^{-1}(\bar{s}_{j;k}d_j)^{-1}\{\ \dots\ \}$"
  $\to$ "$r_{m;l}(\bar{s}_{j;k}d_j^{-1})r_{m;l}^{-1} = \{\ \dots\ \}^{-1}(\bar{s}_{j;k}d_j^{-1})\{\ \dots\ \}$".

The last error is a typo. The first three errors are due to a mistake of [1, Lemma 5.9]. To be precise, ε in Lemma 5.9 is correct as follows.

$$\varepsilon = \begin{cases} 2 & \text{for Relation (D1e)}' \text{ for } m = i+1, \\ 1 & \text{for Relation (D1d)}' \text{ for } m = i+1, \\ -1 & \text{for Relations (D1b)}' \text{ for } i = 2,\ \text{(D1c)}' \text{ for } i = 4, \\ & \ \text{and (D1d)}' \text{ for } m = i-1 \\ -2 & \text{for Relation (D1e)}' \text{ for } m = i, \\ 0 & \text{for the other cases.} \end{cases}$$

The relation $[r_{2;1}] = [r_{1;1}]$ in $H_1(\mathcal{M}(N_{g,n}); \mathbb{Z})$. By (the corrected version of) Relation (D2b) for $i = 2$, we have $2[r_{1,k}] = 1$ for $1 \leq k \leq n-1$. By Relation (D2a) for $m = i-1$ or $m = i$, we have $[r_{i+1,k}] = -[r_{i,k}]$ for $1 \leq i \leq g-1$. Hence, by using these relations, we have

$$[r_{2,k}] = -[r_{1,k}] = [r_{1,k}]$$

[1] There is a place where d_k is written incorrectly as d_{n-1}.

in $H_1(\mathcal{M}(N_{g,n});\mathbb{Z})$. Since $[r_{2,k}]=[r_{1,k}]$ has order 2 in $H_1(\mathcal{M}(N_{g,n});\mathbb{Z})$, by a similar argument above, we also have $[r_{i+1,k}]=[r_{i,k}]$ in $H_1(\mathcal{M}(N_{g,n});\mathbb{Z})$ for $1\leq i\leq g-1$.

Relations (D1a)–(D4g) for $n=2$. For $1\leq k\leq n-1$, we regard $N_{g,k}$ as the surface which is obtained from $N_{g,n}$ by attaching $n-k$ disks along $\delta_{n-1},\delta_{n-2},\dots,\delta_k$. The natural inclusion $N_{g,k+1}\hookrightarrow N_{g,k}$ induces a surjective homomorpism $\mathcal{M}(N_{g,k+1})\to\mathcal{M}^+(N_{g,k},x_0)$, where x_0 is a base point in the interior of the disk capping δ_k. Since the kernel of this homomorphism is the infinite cyclic group generated by the boundary twist d_{k+1}, a relation in $\mathcal{M}^+(N_{g,k},x_0)$ lifts to $\mathcal{M}(N_{g,k+1})$ up to a power of d_{k+1}. We can regard a relation in $\mathcal{M}(N_{g,k+1})$ as a relation $\mathcal{M}(N_{g,n})$ by considering an inclusion $N_{g,k+1}\hookrightarrow N_{g,n}$ as in [1, Figure 21].

We have the following exact sequence:

$$1\longrightarrow\pi_1(N_{g,k})^+\xrightarrow{j_{x_0}}\mathcal{M}^+(N_{g,k},x_0)\xrightarrow{\mathcal{F}}\mathcal{M}(N_{g,k})\longrightarrow 1,$$

where j_{x_0} is the point pushing homomorphism about the base point x_0 and $\mathcal{F}$ is the forgetful homomorphism induced by forgetting x_0. By the exactness, for $\gamma\in\pi_1(N_{g,k})^+$ and $f\in\mathcal{M}^+(N_{g,k},x_0)$, the product $fj_{x_0}(\gamma)f^{-1}$ lies in the image $j_{x_0}(\pi_1(N_{g,k})^+)$. Relations (D1a)–(D4g) are induced from conjugation of images of generators for $\pi_1(N_{g,k})^+$ in [1, Lemma 5.5] by lifts of generators of $\mathcal{M}(N_{g,k})$ with respect to $\mathcal{F}$.

We explain the meaning of subscripts in Relations (D1a)–(D4g).

- The subscripts $1\leq j,k\leq n-1$ and $1\leq i\leq g$ in Relations (D1a)–(D4g): they are ones of the images by j_{x_0} of generators for $\pi_1(N_{g,k})^+$ in [1, Lemma 5.5], that lies in the left-hand side in Relations (D1a)–(D4g), which are conjugated by lifts of generators of $\mathcal{M}(N_{g,k})$ with respect to $\mathcal{F}$.
- The subscripts $1\leq l,t\leq k$ and $1\leq m\leq g$ in Relations (D1a)–(D4g): they are ones of the lifts of generators for $\mathcal{M}(N_{g,k})$ with respect to $\mathcal{F}$, that also lies in the left-hand side in Relations (D1a)–(D4g), which conjugate $j_{x_0}(\gamma)$'s, where γ is a generator of $\pi_1(N_{g,k})^+$ in [1, Lemma 5.5].

When $n=2$, since $k\leq n-1=1$, we can not take $1\leq l,t<k$. Hence the presentation in [1, Proposition 3.2] does not have Relations (D$*$d)–(D$*$g). Similarly, since we have $k=j$ when $n=2$, elements $s_{j,k}$ and $\bar{s}_{j,k}$ are not defined. Hence the presentation in [1, Proposition 3.2] does not have Relations (D3$*$) and (D4$*$), either. Thus, when $n=2$, Relations (D1a)–(D1c) and (D2a)–(D2c) only remain. In particular, in the case $g\leq 3$ and $n=2$, since b is not defined in $\mathcal{M}(N_{g,k})$, Relations (D1a), (D1b), (D2a), and (D2b) only remain.

The abelianization of $\mathcal{M}(N_{1,n})$. First, we remark that Relations (D2e)–(D4g) of [1, Lemma 5.5] are trivial relations in $H_1(\mathcal{M}(N_{g,n});\mathbb{Z})$, since the relations form $YXY^{-1}=ZXZ^{-1}$. Since the elements a_i, y, b, and $a_{i;j}$ are not defined in $\mathcal{M}(N_{1,n})$, in the case $n=1$, the presentation in [1, Proposition 3.2] does not have Relations (D$*$a)–(D$*$d) and (D1$*$). Thus, we have the presentation for $\mathcal{M}(N_{1,n})$ in [1, Proposition 3.2] is as follows:

Proposition 1. *For $n\geq 2$, $\mathcal{M}(N_{1,n})$ admits the presentation with generators d_i $(1\leq i\leq n-1)$, $r_{1,j}$ $(1\leq j\leq n-1)$, $s_{i,j}$ $(1\leq i<j\leq n-1)$, and $\bar{s}_{i,j}$ $(1\leq i<j\leq n-1)$, and the following defining relations:*

Relations (D0), (D2e), (D2f), (D2g), (D3e), (D3f), (D3g), (D4e), (D4f), and (D4g) in [1, Proposition 3.2].

Since Relations (D2e)–(D4g) of [1, Lemma 5.5] are trivial relations in $H_1(\mathcal{M}(N_{g,n});\mathbb{Z})$, by Proposition 1, the abelianization of $\mathcal{M}(N_{1,n})$ is as follows:

Proposition 2. *For $n \geq 2$,*

$$H_1(\mathcal{M}(N_{1,n});\mathbb{Z}) \cong \mathbb{Z}^{2\left(n-1+\binom{n-1}{2}\right)}.$$

The abelianization of $\mathcal{M}(N_{2,n})$. Since the element b is not defined in $\mathcal{M}(N_{2,n})$, in the case $n=2$, the presentation in [1, Proposition 3.2] does not have Relations (D∗c). Thus, we have the presentation for $\mathcal{M}(N_{2,n})$ in [1, Proposition 3.2] is as follows:

Proposition 3. *For $n \geq 2$, $\mathcal{M}(N_{2,n})$ admits the presentation with generators a_1, y, d_i $(1 \leq i \leq n-1)$, $a_{1;j}$ $(1 \leq j \leq n-1)$, $r_{i,j}$ $(1 \leq i \leq 2,\ 1 \leq j \leq n-1)$, $s_{i,j}$ $(1 \leq i < j \leq n-1)$, and $\bar{s}_{i,j}$ $(1 \leq i < j \leq n-1)$, and the following defining relations for $1 \leq k \leq n-1$, $1 \leq l < t < k$, and $1 \leq i, m \leq g$:*

(0) $ya_1y^{-1} = a_1^{-1}$,

(D1a) $a_1(a_{1;k}a_1^{-1})a_1^{-1} = a_{1;k}a_1^{-1}$,

(D1b) $y(a_{1;k}a_1^{-1})y^{-1} = (a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k}$,

(D1d) $a_{1;l}(a_{1;k}a_1^{-1})a_{1;l}^{-1} = \{(s_{l,k}d_l^{-1})(a_{1;k}a_1^{-1})\}^{-1}(a_{1;k}a_1^{-1})\{(s_{l,k}d_l^{-1})(a_{1;k}a_i^{-1})\}$,

(D1e) $r_{m;l}(a_{1;k}a_1^{-1})r_{m;l}^{-1} =$
$$\begin{cases} \{r_{1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{1;k}\}(s_{l,k}d_l^{-1})(a_{1;k}a_1^{-1}) \\ (\bar{s}_{l,k;1}d_l^{-1})^{-1}\{r_{1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{1;k}\}^{-1}d_k^{-2} & \text{for } m=1, \\ r_{2;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{2;k}(\bar{s}_{l,k;2}d_l^{-1})(a_{2;k}a_1^{-1})d_k^2 & \text{for } m=2, \end{cases}$$

(D1f) $s_{l,t}(a_{1;k}a_1^{-1})s_{l,t}^{-1} = a_{1;k}a_1^{-1}$,

(D1g) $\bar{s}_{l,t}(a_{1;k}a_1^{-1})\bar{s}_{l,t}^{-1} = [(\bar{s}_{l,k}d_l^{-1})^{-1}, s_{t,k}d_t^{-1}]^{-1}(s_{l,k}d_l^{-1})(a_{1;k}a_1^{-1})$
$r_{1;k}(\bar{s}_{t,k}d_t^{-1})r_{1;k}^{-1}(s_{l,k}d_l^{-1})^{-1}r_{1;k}(\bar{s}_{t,k}d_t^{-1})^{-1}r_{1;k}^{-1}[(\bar{s}_{l,k}d_l^{-1})^{-1}, (s_{t,k}d_t^{-1})]$,

(D2a) $a_1r_{i;k}a_1^{-1} = \begin{cases} r_{2;k}r_{1;k}(a_{1;k}a_1^{-1})^{-1}r_{2;k}(a_{1;k}a_1^{-1}) & \text{for } i=2, \\ (a_{1;k}a_1^{-1})^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1}) & \text{for } i=1, \end{cases}$

(D2b) $yr_{i;k}y^{-1} = \begin{cases} \{(a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k}\}^{-1}r_{1;k}^{-1}\{(a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k}\} & \text{for } i=1, \\ (a_{1;k}a_1^{-1})r_{1;k}(a_{1;k}a_1^{-1})^{-1}r_{2;k}r_{1;k} & \text{for } i=2, \end{cases}$

(D2d) $a_{1;l}r_{i;k}a_{1;l}^{-1} =$
$$\begin{cases} \{(s_{l,k}d_l^{-1})(a_{1;k}a_1^{-1})\}^{-1}(a_{1;k}a_1^{-1})(s_{l,k}d_l^{-1})r_{2;k} \\ r_{1;k}(a_{1;k}a_1^{-1})^{-1}r_{2;k}(\bar{s}_{l,k;2}d_l^{-1})\{(s_{l,k}d_l^{-1})(a_{1;k}a_1^{-1})\} & \text{for } i=2, \\ (a_{1;k}a_1^{-1})^{-1}(s_{l,k}d_l^{-1})^{-1}r_{2;k}^{-1}(a_{1;k}a_1^{-1})(\bar{s}_{l,k;1}d_l^{-1})^{-1} & \text{for } i=1, \end{cases}$$

and Relations (D0), (D2e)–(D3b), (D3d)–(D4b), and (D4d)–(D4g) in [1, Proposition 3.2],

where $\bar{s}_{l,k;1} = \bar{s}_{l,k}$ and $\bar{s}_{l,k;2} = \{(a_{1;k}a_1^{-1})^{-1}r_{2;k}\}^{-1}\bar{s}_{l,k}\{(a_{1;k}a_1^{-1})^{-1}r_{2;k}\}$.

By using Proposition 3, we have the following proposition.

Proposition 4.

$$H_1(\mathcal{M}(N_{2,n});\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2^3 & \text{for } n=2, \\ \mathbb{Z}^{\binom{n-1}{2}+1} \oplus \mathbb{Z}_2^{2(n-1)+1} & \text{for } n \geq 3. \end{cases}$$

Proof. For conveniences, we write an element of $\mathcal{M}(N_{2,n})$ and its equivalence class in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$ by the same symbol. We recall that Relations (D0), (D2e)–(D3b), (D3d)–(D4b), and (D4d)–(D4g) are trivial in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$ again. The last relations mean that $\bar{s}_{l,k;i}=\bar{s}_{l,k}$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$. Relations (D1a), (D1d), (D1f), and (D1g) are trivial in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$. Relation (0) is equivalent to the relation $a_1^2=1$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$. By an argument in the second paragraph (and an easy argument), Relations (D2a) and (D2b) are equivalent to the relations $r_{1;k}=r_{2;k}$ and $r_{1;k}^2=1$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$. Up to the relations $a_1^2=1$, $r_{1;k}=r_{2;k}$, and $r_{1;k}^2=1$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$, Relation (D1b) for $i=1$ is equivalent to the relation $a_{1;k}^2=1$, and Relation (D1b) for $i=2$ is equivalent to the relation $r_{1;k}=d_k$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$.

In the case $n=2$, by an argument in the paragraph "Relations (D1a)–(D4g) for $n=2$", only Relations (D1a), (D1b), (D2a), and (D2b) remain in the presentation of Proposition 3. Hence, as a presentation of an abelian subgroup (i.e. we omit commutative relations), we have

$$\begin{aligned}
& H_1(\mathcal{M}(N_{2,2});\mathbb{Z}) \\
\cong\ & \langle a_1, y, d_1, a_{1;1}, r_{1;1} \,|\, a_1^2=a_{1;1}^2=r_1^2=1, r_{1,1}=d_1\rangle \\
\cong\ & \langle a_1, y, a_{1;1}, r_{1;1} \,|\, a_1^2=a_{1;1}^2=r_1^2=1\rangle \\
\cong\ & \mathbb{Z}\oplus\mathbb{Z}_2^3.
\end{aligned}$$

Assume that $n\geq 3$. We recall that $\bar{s}_{l,k;i}=\bar{s}_{l,k}$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$. Up to the relations $r_{1;k}=r_{2;k}$ and $r_{1;k}^2=1$, Relation (D1e) for $m\in\{1,2\}$ is equivalent to the relation $s_{l;k}\bar{s}_{l;k}d_k^{-2}=1$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$. Similarly, Relation (D2d) for $i\in\{1,2\}$ is equivalent to the relation $s_{l;k}\bar{s}_{l;k}d_l^{-2}=1$ in $H_1(\mathcal{M}(N_{2,n});\mathbb{Z})$. Thus, we have

$$\begin{aligned}
& H_1(\mathcal{M}(N_{2,n});\mathbb{Z}) \\
\cong\ & \langle a_1, y, d_i, a_{1;i}, r_{1;i}\ (1\leq i\leq n-1), s_{i,j}, \bar{s}_{i,j}\ (1\leq i<j\leq n-1) \\
& \quad |\, a_1^2=a_{1;i}^2=r_{1;i}^2=r_{1,i}d_i^{-1}=1\ (1\leq i\leq n-1), \\
& \quad s_{i;j}\bar{s}_{i;j}d_j^{-2}=s_{i;j}\bar{s}_{i;j}d_i^{-2}=1\ (1\leq i<j\leq n-1)\rangle \\
\cong\ & \langle a_1, y, a_{1;i}, r_{1;i}\ (1\leq i\leq n-1), s_{i,j}, \bar{s}_{i,j}\ (1\leq i<j\leq n-1) \\
& \quad |\, a_1^2=a_{1;i}^2=r_{1;i}^2=1\ (1\leq i\leq n-1), \\
& \quad s_{i;j}\bar{s}_{i;j}r_{1;j}^{-2}=s_{i;j}\bar{s}_{i;j}r_{1;i}^{-2}=1\ (1\leq i<j\leq n-1)\rangle \\
\cong\ & \langle a_1, y, a_{1;i}, r_{1;i}\ (1\leq i\leq n-1), s_{i,j}, \bar{s}_{i,j}\ (1\leq i<j\leq n-1) \\
& \quad |\, a_1^2=a_{1;i}^2=r_{1;i}^2=1\ (1\leq i\leq n-1),\ s_{i;j}\bar{s}_{i;j}=s_{i;j}\bar{s}_{i;j}=1\ (1\leq i<j\leq n-1)\rangle \\
\cong\ & \langle a_1, y, a_{1;i}, r_{1;i}\ (1\leq i\leq n-1), s_{i,j}, \bar{s}_{i,j}\ (1\leq i<j\leq n-1) \\
& \quad |\, a_1^2=a_{1;i}^2=r_{1;i}^2=1\ (1\leq i\leq n-1),\ s_{i;j}\bar{s}_{i;j}=1\ (1\leq i<j\leq n-1)\rangle \\
\cong\ & \langle a_1, y, a_{1;i}, r_{1;i}\ (1\leq i\leq n-1), s_{i,j}\ (1\leq i<j\leq n-1) \\
& \quad |\, a_1^2=a_{1;i}^2=r_{1;i}^2=1\ (1\leq i\leq n-1)\rangle \\
\cong\ & \mathbb{Z}^{\binom{n-1}{2}+1}\oplus\mathbb{Z}_2^{2(n-1)+1}.
\end{aligned}$$

□

References

[1] R. Kobayashi, G. Omori, *An infinite presentation for the mapping class group of a non-orientable surface with boundary*, Osaka J. Math. **59** (2022), no. 2, 269–314.

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